

Static Optimization with MATLAB

Day 2: Constrained Optimization with Inequality Constraints

GSEFM Pre-Semester Course

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Problem Setting I

Definition (Optimization Problem with Inequality Constraints)

An optimization problem with equality constraints consists of a set of feasible solutions M , given as:

$$M = \{x \in \mathbb{R}^n \mid g(x) \leq 0\},$$

where $g : \mathbb{R}^n \rightarrow \mathbb{R}^l$, and an objective function $f : M \rightarrow \mathbb{R}$. The goal is to find the maximum value of $f(x)$ over all $x \in M$, which is expressed as:

$$\max_{x \in M} f(x)$$

Problem Setting II

- We assume that f and the equality constraints $g_1, \dots, g_l : \mathbb{R}^n \rightarrow \mathbb{R}$ are twice continuously differentiable.
- Basic idea: if the constraint is binding, the problem is the same as for equality constraints, otherwise we are back to the case without constraint.

The Kuhn-Tucker Problem I

We can also set up a Lagrangian for this problem:

$$\mathcal{L}(x, \lambda) = f(x) - \lambda' g(x)$$

where λ is a vector of multipliers.

The Kuhn-Tucker Problem II

Definition (Kuhn-Tucker Condition)

The following conditions are called Kuhn-Tucker conditions:

$$D_x \mathcal{L}(x, \lambda) = Df(x) - \lambda' Dg(x) = \mathbf{0}$$

for each $j = 1, \dots, l$ $g_j(x) \leq 0$

$\lambda_j \geq 0$ non-negativity of LM

$\lambda_j g_j(x) = 0$ complementary slackness condition

Complementary Slackness Conditions

The complementary slackness condition captures that either the constraint is binding ($g_j(x) = 0$) or the constraint is slack ($g_j(x) < 0$), but then the associated multiplier is zero ($\lambda_j = 0$):

- If $\lambda_j > 0$, it has to be that $g_j(x) = 0$
 $\Rightarrow$ Interpretation: If the multiplier is positive, the agent benefits from one unit more of the resource $\Rightarrow$ The constraint must be binding.
- If $g_j(x) < 0$, it has to be that $\lambda_j = 0$
 $\Rightarrow$ Interpretation: If the constraint is not binding, the marginal utility from a relaxation of the constraint is zero, i.e. its shadow price is zero.

Partitioning the constraints I

Now we renumber the constraint functions $g_j(x)$, $j = 1, \dots, I$ such that the binding constraints come first, for x^* being a feasible point:

$$g_j(x^*) = 0 \text{ for } j = 1, 2, \dots, B \quad (\text{binding constraints})$$

$$g_j(x^*) < 0 \text{ for } j = B + 1, \dots, I \quad (\text{nonbinding constraints})$$

and define vectors of constraints

- active at x^* : $g^b(x^*) = [g_1(x^*), \dots, g_B(x^*)]'$
- inactive at x^* : $g^n(x^*) = [g_{B+1}(x^*), \dots, g_I(x^*)]'$.

Accordingly, we can partition the vector of multipliers, yielding:

$$g(x^*) = \begin{bmatrix} g^b(x^*) \\ g^n(x^*) \end{bmatrix} \quad \lambda = (\lambda_b, \lambda_n)'$$

Partitioning the constraints II

Using this notation, the Lagrange condition can be written as

$$Df(x) - (\lambda_b, \lambda_n) \begin{bmatrix} Dg^b(x) \\ Dg^n(x) \end{bmatrix} = Df(x) - \lambda'_b Dg^b(x) - \lambda'_n Dg^n(x) = \mathbf{0}$$

The rank condition for the Jacobian of the constraints now only applies to the *binding constraints*, i.e.

$$\text{rank } Dg^b(x) = B$$

It can be interpreted almost exactly as in the Lagrange case, except that it now applies only to the constraints that are binding at an optimal solution of the problem.

Partitioning the constraints III

We further partition the binding constraints g^b into those with positive Lagrange Multiplier ($\lambda_j > 0$) and those with zero Lagrange Multiplier ($\lambda_j = 0$). We call the vector of constraints with positive Lagrange Multiplier $g^{b,>}$. Note that for these inequalities the so called strict complementary slackness (SCSC) condition holds:

$$\lambda_j + g_j(x^*) \neq 0.$$

The Kuhn-Tucker Theorem

Theorem (The Kuhn-Tucker Theorem)

Let x^ be an optimal solution to the Kuhn-Tucker problem*

$$\max_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad g(x) \leq \mathbf{0}$$

where f and g are continuously differentiable and $\text{rank } Dg^b(x^) = B < n$. Then there exist non-negative Lagrange multipliers $\lambda^* \in \mathbb{R}_+^I$ such that (x^*, λ^*) is a KT-point.*

Sufficient Conditions for a Strict Local Maximum I

Theorem (Sufficient condition for a strict local maximum)

Let x^* be a feasible point satisfying the Kuhn-Tucker condition for some λ^* . Assume that the Hessian of the Lagrangian function with respect to the choice variables x , evaluated at (x^*, λ^*) , given by

$$D_x^2 \mathcal{L}(x^*, \lambda^*) = D^2 f(x^*) - \lambda^{*'} D^2 h(x^*),$$

is negative definite subject to the constraints $Dg^{b,>}(x^*)y = \mathbf{0}$, i.e., it holds that

$$y' D_x^2 \mathcal{L}(x^*, \lambda^*) y < 0, \text{ for all } y \in \mathbb{R}^n, y \neq 0 \text{ with } Dg^{b,>}(x^*)y = \mathbf{0}.$$

Then x^* is a strict local maximizer of f subject to $g(x) \leq \mathbf{0}$.

Sufficient Conditions for a Strict Local Maximum II

⇒ To check this condition we can use, as before, the bordered Hessian with the binding constraints with positive Lagrange Multiplier $g^{b,>}$ (before that we used the equality constraints h). Now, we have to look at the $n - B^>$ leading principal minors, where $B^> \leq B$ is the dimension of $g^{b,>}$.

Extension to Generalized Concave Problems I

Theorem (Sufficient Condition for a Local Maximum)

Assume that the objective function of the constraint problem f is pseudoconcave ► Definition and the constraint functions g_j are all quasiconvex ► Definition (i.e. the constraint set is convex). Let (x^, λ^*) be a pair of vectors that satisfy the necessary conditions given in the Kuhn-Tucker Theorem (Lagrange + complementary slackness conditions). Then x^* is an optimal solution of the constrained optimization problem.*

Extension to Generalized Concave Problems II

Theorem (Uniqueness of the Global Maximum)

Let x^ be an optimal solution of the constrained maximization problem. If f is strictly quasiconcave ► Definition and the constraint functions g_j are all quasiconvex ► Definition, then x^* is the only optimal solution of the original problem.*

Example

Maximize $f(x) = x_1 x_2$ subject to $x_1^2 + x_2^2 \leq 1$. Set up the Lagrangian:

$$\mathcal{L}(x_1, x_2, \lambda) = x_1 x_2 - \lambda[x_1^2 + x_2^2 - 1]$$

FOCs:

$$\frac{\partial \mathcal{L}}{\partial x_1}(x_1^*, x_2^*, \lambda^*) = x_2^* - 2\lambda^* x_1^* = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2}(x_1^*, x_2^*, \lambda^*) = x_1^* - 2\lambda^* x_2^* = 0$$

$$\lambda^*(x_1^{*2} + x_2^{*2} - 1) = 0$$

$$\lambda^* \geq 0$$

$$x_1^{*2} + x_2^{*2} \leq 1$$

Example Solution I

Solving the first two equations for λ^* :

$$\lambda^* = \frac{x_2^*}{2x_1^*} = \frac{x_1^*}{2x_2^*} \quad \Leftrightarrow \quad x_1^{*2} = x_2^{*2}$$

Go through possible cases:

1. $x_1^{*2} + x_2^{*2} < 1$: $\Rightarrow \lambda^* = 0 \Rightarrow$ then by both FOCs: $x_1^* = x_2^* = 0$
2. $\lambda^* > 0$: $\Rightarrow$ inequality constraint becomes binding: $x_1^{*2} + x_2^{*2} = 1$ plugging in the condition from above, gives:

$$x_1^{*2} = x_2^{*2} = \frac{1}{2}$$

Example Solution II

This gives 4 candidate solutions:

$x_1^* = +\frac{1}{\sqrt{2}},$	$x_2^* = +\frac{1}{\sqrt{2}},$	$\lambda^* = +\frac{1}{2}$
$x_1^* = -\frac{1}{\sqrt{2}},$	$x_2^* = -\frac{1}{\sqrt{2}},$	$\lambda^* = +\frac{1}{2}$
$x_1^* = +\frac{1}{\sqrt{2}},$	$x_2^* = -\frac{1}{\sqrt{2}},$	$\lambda^* = -\frac{1}{2}$
$x_1^* = -\frac{1}{\sqrt{2}},$	$x_2^* = +\frac{1}{\sqrt{2}},$	$\lambda^* = -\frac{1}{2}$

- we can discard the last two because they imply $\lambda^* < 0$ (they are actually minimizing $x_1 x_2$ on the constraint set $x_1^2 + x_2^2 \leq 1$)

Example Solution III

Lastly, we can plug in the remaining three candidates into objective function:

1. $(x_1^* = x_2^* = 0, \lambda^* = 0)$: $\Rightarrow f(0, 0) = 0$
2. $(x_1^* = x_2^* = \frac{1}{\sqrt{2}}, \lambda^* = \frac{1}{2})$: $\Rightarrow f(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = \frac{1}{2}$
3. $(x_1^* = x_2^* = -\frac{1}{\sqrt{2}}, \lambda^* = \frac{1}{2})$: $\Rightarrow f(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = \frac{1}{2}$

$\Rightarrow$ The last two points might be maximands of our problem. Is the constraint

qualification at the maximum points satisfied (only for 2. and 3., because at 1. the constraint do not bind):

$$Dg(x_1^*, x_2^*) = [2x_1^* \ 2x_2^*] = ([\sqrt{2} \ \sqrt{2}], [-\sqrt{2} \ -\sqrt{2}])$$

which are both unequal to zero, so NDCQ is satisfied.

Example Solution IV

Is $(x_1^* = x_2^* = 0, \lambda^* = 0)$ a maximum? As no constraint is binding we only have to look at the "normal" Hessian:

$$H = \begin{bmatrix} -2\lambda^* & 1 \\ 1 & -2\lambda^* \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

First leading principal minor is 0. Therefore H is indefinite. $(x_1^* = x_2^* = 0, \lambda^* = 0)$ is not a maximum (we can see in the graph that it is a saddle point).

Example Solution V

Are $(x_1^* = x_2^* = \pm \frac{1}{\sqrt{2}}, \lambda^* = \frac{1}{2})$ sufficient solution (i.e. really maxima)? With

Kuhn-Tucker, we only have to include **binding** constraints in the bordered Hessian:

As $\lambda^* > 0$ for both points, have to include it: First write general bordered Hessian for problem, then assess its properties at the optimum points found above:

$$\bar{H} = \begin{bmatrix} 0 & 2x_1^* & 2x_2^* \\ 2x_1^* & -2\lambda^* & 1 \\ 2x_2^* & 1 & -2\lambda^* \end{bmatrix}$$

This yields: for $|\bar{H}_3| = 8x_1^{*2}\lambda^* + 8x_1^*x_2^* + 8x_2^{*2}\lambda^*$

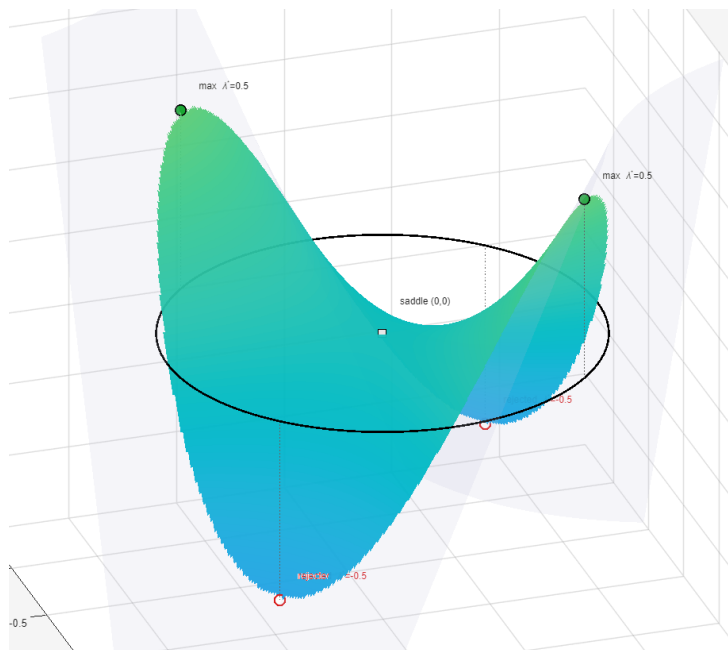
Example Solution VI

Plug in the optimal values:

- $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{2}) : \Rightarrow |\bar{H}_3| = 8$
- $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{2}) : \Rightarrow |\bar{H}_3| = 8$
- For a maximum, need that $|\bar{H}_3|$ has the sign of $(-1)^{m+1}$, i.e. for this case $(-1)^2 = 1$ which is positive, so indeed we have found two local maxima.

Why is the solution not unique? because $f(x_1, x_2) = x_1 x_2$ is not strictly quasiconcave.

Example Solution VII



Exercise

Solve the problem

$$\begin{aligned} \max_{x_1, x_2} f(x_1, x_2) &= -(x_1 - 4)^2 - (x_2 - 4)^2 \\ \text{s.t. } x_1 + x_2 &\leq 4 \text{ and} \\ x_1 + 3x_2 &\leq 9. \end{aligned}$$

by the Kuhn-Tucker method.

Exercise Solution I

Set up the Lagrangian:

$$\mathcal{L}(x_1, x_2, \lambda) = -(x_1 - 4)^2 - (x_2 - 4)^2 - \lambda_1(x_1 + x_2 - 4) - \lambda_2(x_1 + 3x_2 - 9)$$

FOCs:

$$\frac{\partial \mathcal{L}}{\partial x_1}(x_1^*, x_2^*, \lambda^*) = -2(x_1^* - 4) - \lambda_1^* - \lambda_2^* = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2}(x_1^*, x_2^*, \lambda^*) = -2(x_2^* - 4) - \lambda_1^* - 3\lambda_2^* = 0$$

$$\lambda_1^*(x_1^* + x_2^* - 4) = 0$$

$$\lambda_2^*(x_1^* + 3x_2^* - 9) = 0$$

$$\lambda_1^*, \lambda_2^* \geq 0$$

$$x_1^* + x_2^* \leq 4$$

$$x_1^* + 3x_2^* \leq 9$$

Exercise Solution II

To solve this system of condition we have to consider all possibilities about the values of λ_i for $i \in \{1, 2\}$.

We have to consider the following 4 cases:

1. $g^1, g^2 < 0 \Rightarrow \lambda_1, \lambda_2 = 0$
2. $\lambda_1 > 0, g^2 < 0 \Rightarrow g^1 = 0, \lambda_2 = 0$
3. $g^1 < 0, \lambda_2 > 0 \Rightarrow \lambda_1 = 0, g^2 = 0$
4. $\lambda_1 > 0, \lambda_2 > 0 \Rightarrow g^1 = 0, g^2 = 0$.

Exercise Solution III

Case 1: $g^1, g^2 > 0 \Rightarrow \lambda_1, \lambda_2 = 0$

The Kuhn-Tucker conditions are

$$-2(x_1^* - 4) = 0,$$

$$-2(x_2^* - 4) = 0,$$

$$x_1^* + x_2^* < 4,$$

$$x_1^* + 3x_2^* < 9$$

$$\lambda_1^*, \lambda_2^* = 0$$

If you solve the first two equations for x_1^* and x_2^* you get $x_1^* = 4$ and $x_2^* = 4$. However, this solution violates the two inequalities.

Exercise Solution IV

Case 2: $\lambda_1 > 0, g^2 < 0 \Rightarrow g^1 = 0, \lambda_2 = 0$

The Kuhn-Tucker conditions are

$$-2(x_1^* - 4) - \lambda_1^* = 0,$$

$$-2(x_2^* - 4) - \lambda_1^* = 0,$$

$$x_1^* + x_2^* - 4 = 0,$$

$$x_1^* + 3x_2^* < 9$$

$$\lambda_1^* > 0$$

$$\lambda_2^* = 0.$$

From the first two equations we get $x_1^* = x_2^*$.

Using the third equation we get $x_1^* = x_2^* = 2$ and $\lambda_1^* = 4$.

It is a solution because all inequalities and equalities are satisfied.

Exercise Solution V

Case 3: $g^1 < 0, \lambda_2 > 0 \Rightarrow \lambda_1 = 0, g^2 = 0$

The Kuhn-Tucker conditions are

$$-2(x_1^* - 4) - \lambda_2^* = 0, \quad (1)$$

$$-2(x_2^* - 4) - 3\lambda_2^* = 0, \quad (2)$$

$$x_1^* + x_2^* < 4, \quad (3)$$

$$x_1^* + 3x_2^* - 9 = 0, \quad (4)$$

$$\lambda_1^* = 0 \quad (5)$$

$$\lambda_2^* > 0. \quad (6)$$

From the equations (1) and (2) we get $x_2^* = 3x_1^* - 8$.

Using equation (4) we get $x_1^* = 3.3$ (and $x_2^* = 1.9$).

It is not a solution because it does not satisfy equation (3).

Exercise Solution VI

Case 4: $\lambda_1 > 0, \lambda_2 > 0 \Rightarrow g^1 = 0, g^2 = 0,$

The Kuhn-Tucker conditions are

$$-2(x_1^* - 4) - \lambda_1 - \lambda_2 = 0, \quad (7)$$

$$-2(x_2^* - 4) - \lambda_1 - 3\lambda_2 = 0, \quad (8)$$

$$x_1^* + x_2^* - 4 = 0, \quad (9)$$

$$x_1^* + 3x_2^* - 9 = 0, \quad (10)$$

$$\lambda_1 > 0 \quad (11)$$

$$\lambda_2 > 0. \quad (12)$$

From equations (9) and (10) we get $x_1^* = 1.5$ and $x_2^* = 2.5$.

Replacing in equation (9) and (10) into equations (7) and (8) we get $\lambda_1^* = 6$ and $\lambda_2^* = -1$.

This is not a solution because it violates the condition $\lambda_2^* \geq 0$.

Exercise Solution VII

Is the constraint qualification at the maximum point satisfied (as $\lambda_1^* > 0$ and $\lambda_2 = 0$ for the point $(x_1^* = 2, x_2^* = 2)$, we have to look only at the first constraint:):

$$Dg^1(x_1^*, x_2^*) = [1 \ 1]$$

which are unequal to zero, so NDCQ is satisfied.

Exercise Solution VIII

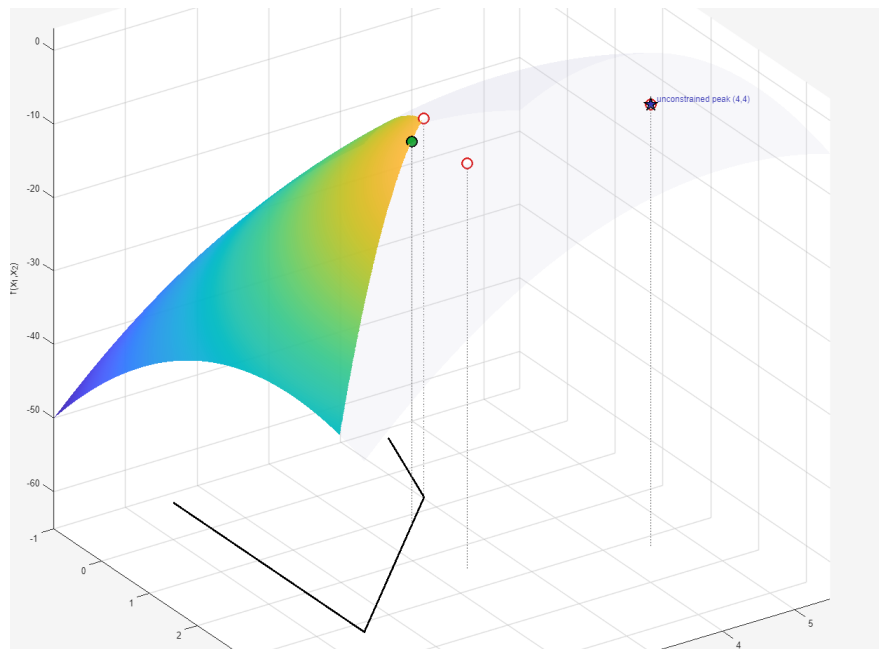
Is solution sufficient (i.e. really a maximum)?

With Kuhn-Tucker, we only have to include binding constraints in the bordered Hessian as well:

$$\bar{H} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

This yields for $|\bar{H}_3| = 4 > 0$. Since $(-1)^{m+1}$ is also positive for $m = 1$, $(x_1^* = 2, x_2^* = 2)$ is indeed a maximum.

Exercise Solution IX



References I

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Pseudoconcave Function I

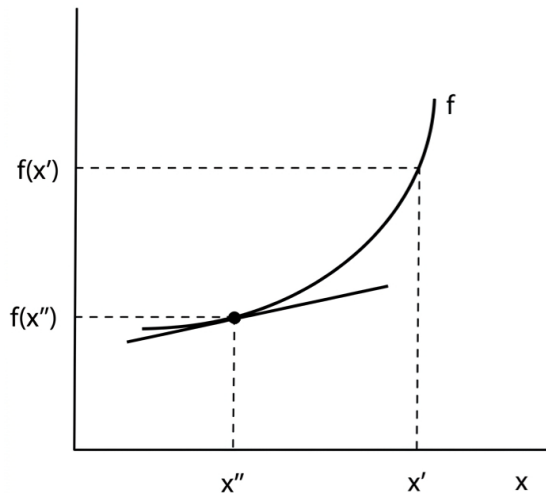
Definition (Pseudoconcave Function)

A continuously differentiable function f is pseudoconcave in a set M if, given any two points x and y in X , we have that

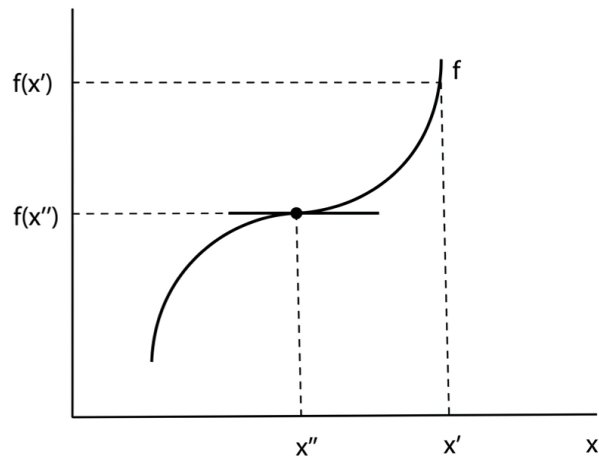
$$f(x) > f(y) \Rightarrow Df(y)(x - y) > 0$$

concave $\Rightarrow$ pseudoconcave $\Rightarrow$ strictly quasiconcave $\Rightarrow$ quasiconcave (for a differentiable function)

Pseudoconcave Function II



(a) f is strictly quasiconcave $Df(x'')(x' - x'') > 0$



(b) f is quasiconcave, but not strictly $Df(x'')(x' - x'') = 0$

Figure: Pseudoconcave function

Quasiconvex Function

Definition (Quasiconvex Function)

Let f be a real-valued function defined on a convex set M . f is quasiconvex if for all x and y in M and all $\lambda \in [0, 1]$ we have

$$f[(1 - \lambda)x + \lambda y] \leq \max\{f(x), f(y)\}$$

convex $\Rightarrow$ pseudoconvex $\Rightarrow$ strictly quasiconvex $\Rightarrow$ quasiconvex (for a differentiable function) [▶ Go Back](#) [▶ Go Back](#)

Strictly Quasiconcave Function

Definition (Strictly Quasiconcave Function)

Let f be a real-valued function defined on a convex set M . f is strictly quasiconcave if for all x and y in M and all $\lambda \in (0, 1)$ we have

$$f[(1 - \lambda)x + \lambda y] > \min\{f(x), f(y)\}$$

concave $\Rightarrow$ pseudoconcave $\Rightarrow$ strictly quasiconcave $\Rightarrow$ quasiconcave (for a differentiable function) [» Go Back](#)